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On Two Entire Functions Which Together with Their First Derivatives Have the Same Zeros

CHUNG-CHUN YANG

Mathematical Sciences Staff, Naval Research Laboratory, Washington, D.C. 20375

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1. INTRODUCTION

In [2] the author proved the following result. Let

$$p(z) = c_1(z - a_1)^{m_1} (z - a_2)^{m_2} \cdots (z - z_k)^{m_k}$$

and

$$q(z) = c_2(z - a_1)^{n_1} (z - a_2)^{n_2} \cdots (z - a_k)^{n_k}$$

be two nonconstant polynomials where c_1 and c_2 are nonzero constants, m_i and n_i are positive integers, and $a_1, a_2, \ldots, a_k$ are distinct complex numbers. Suppose that $p'(z) = 0 \Leftrightarrow q'(z) = 0$ (without counting the multiplicities). Then

$$m_1/n_1 = m_2/n_2 = \cdots = m_k/n_k = m/n$$

and $p(z) = cq(z)^{m/n}$ (c is a constant $\neq 0$).

It easily can be shown that in general the preceding conclusion will not hold for two transcendental entire functions. However, by imposing some additional conditions on the transcendental entire functions we can obtain a similar result. In fact, we prove the following:

THEOREM. *Let $f(z)$ and $g(z)$ be two transcendental entire functions satisfying the following three conditions:*

(A) *f and g have the same zeros, and all the zeros are simple; (The latter part of the condition can be relaxed somewhat, as can be seen from the final remark of Section 3 of this paper.)*

(B) *f' and g' have the same zeros with the same multiplicities, and*

(C) $\quad \operatorname{Max}\left\{\overline{\lim_{r\to\infty}} \frac{\log\log\log M(r,f)}{\log r}, \overline{\lim_{r\to\infty}} \frac{\log\log\log M(r,g)}{\log r}\right\} = \rho < 1.$

Then f and g must satisfy only one of the following two relations:

(I) $\quad f(z) = cg^k(z),$

where c and k are two constants, and

(II) $\quad f(z) = c_1 e^{\gamma(z)} + c_2, \qquad g(z) = c_3(c_2 e^{-\gamma(z)} + c_1),$

where c_1, c_2, and c_3 are constants and $\gamma(z)$ is an entire function of order less than one.

Remark. Condition (C) cannot be omitted in the theorem. For example, take $f(z) = \exp e^z$ and $g(z) = e^{2z}$. Then clearly f and g satisfy conditions (A) and (B), but they are not related by the relations (I) or (II).

2. Proof of the Theorem

By conditions (A), (B), and (C) we have the following two equations:

$$f(z)/g(z) = e^{\alpha(z)} \tag{1}$$

and

$$f'(z)/g'(z) = e^{\beta(z)}, \tag{2}$$

where $\alpha(z)$ and $\beta(z)$ are entire functions of order less than one. We now differentiate (1) and obtain

$$f'(z) = g'(z)\, e^{\alpha(z)} + \alpha'(z)\, g(z)\, e^{\alpha(z)}. \tag{3}$$

Eliminating f' from this and Eq. (2), we get

$$g'(z)\,(e^{\beta(z)} - e^{\alpha(z)}) = g(z)\, \alpha'(z)\, e^{\alpha(z)}. \tag{4}$$

Since by condition (A) $g(z)$ has only simple zeros, we conclude that the quotient $\alpha'(z)/g'(z)$ must be an entire function. Thus

$$\alpha'(z) = g'(z)\, h(z) \tag{5}$$

for some entire function $h(z)$. Substituting this into (4), we have

$$\begin{aligned} g(z)\, h(z) &= (e^{\beta(z)} - e^{\alpha(z)})\, e^{-\alpha(z)} \\ &= e^{\beta(z)-\alpha(z)} - 1. \end{aligned} \tag{6}$$

Differentiating this, we obtain

$$g'(z)\, h(z) + g(z)\, h'(z) = (\beta(z) - \alpha(z))'\, e^{\beta(z)-\alpha(z)}. \tag{7}$$

Equations (7) and (5) lead to

$$\begin{aligned} g(z)\,h'(z) &= (\beta'(z) - \alpha'(z))\,e^{\beta(z)-\alpha(z)} - g(z)\,h'(z) \\ &= (\beta'(z) - \alpha'(z))\,e^{\beta(z)-\alpha(z)} - \alpha'(z). \end{aligned} \tag{8}$$

There are two cases to be considered: that in which $h'(z) \equiv 0$ and then that in which $h'(z) \not\equiv 0$. Suppose that $h'(z) \equiv 0$, then $h(z) \equiv c$ for some constant $c\ (\neq 0)$. Thus (6) implies

$$g(z) = \frac{e^{\beta(z)-\alpha(z)} - 1}{c}. \tag{9}$$

Also from (5) we have

$$\alpha(z) = cg(z) + a \tag{10}$$

or

$$g(z) = \frac{\alpha(z)}{c} - a \tag{11}$$

for some constant a. It follows that $g(z)$ is of order less than one. This is not consistent with Eq. (9) because there g is either an entire function of order at least one (when $\beta(z) - \alpha(z) \not\equiv$ constant) or a constant (when $\beta(z) - \alpha(z) \equiv$ constant). So only the case that $h'(z) \not\equiv 0$ can hold. Thus from (8) we obtain

$$g(z) = [(\beta'(z) - \alpha'(z))\,e^{\beta(z)-\alpha(z)} - \alpha'(z)]\,h'(z)^{-1}. \tag{12}$$

Again substituting this into (4), we have

$$h'(z)\,g'(z)\,e^{\alpha(z)}(e^{\beta(z)-\alpha(z)} - 1) = [(\beta'(z) - \alpha'(z))\,e^{\beta(z)-\alpha(z)} - \alpha'(z)]\,\alpha'(z)\,e^{\alpha(z)}. \tag{13}$$

There are two subcases here which will be treated separately:

(i) $e^{\beta(z)-\alpha(z)} - 1 \equiv 0$

and

(ii) $e^{\beta(z)-\alpha(z)} - 1 \not\equiv 0.$

In Subcase (i) we conclude from (13) that either $\alpha'(z) \equiv 0$ or $\beta'(z) - \alpha'(z)\,e^{\beta(z)-\alpha(z)} - \alpha'(z) \equiv 0$. The former condition leads to $\alpha(z) \equiv c \equiv$ constant, which gives $f(z) = e^c g(z)$, whereas the latter one leads to

$$\beta'(z) - 2\alpha'(z) \equiv 0. \tag{14}$$

Therefore

$$\beta(z) = 2\alpha(z) + d \tag{15}$$

for some constant d.

Combining this with Eqs. (1) and (2), we have

$$c^d(f(z)/g(z))^2 = f'(z)/g'(z). \tag{16}$$

Hence

$$f'(z)/f^2(z) = d_0[g'(z)/g^2(z)] \tag{17}$$

where $d_0 = e^d$. Integrating (17), we obtain

$$1/f(z) = [d_0/g(z)] + d_1\,, \tag{18}$$

where d_1 is a constant $\neq 0$. This gives

$$d_1 f(z)\, g(z) + d_0 f(z) - g(z) = 0 \tag{19}$$

or

$$(d_1 f(z) - 1)\,(g(z) + d_0/d_1) = -d_0/d_1\,. \tag{20}$$

Thus we have

$$d_1 f(z) - 1 = e^{\mu_1(z)} \tag{21}$$

and

$$g(z) + \frac{d_0}{d_1} = e^{\mu_2(z)}, \tag{22}$$

where $\mu_1(z)$ and $\mu_2(z)$ are entire functions of order less than one which satisfy

$$\mu_1(z) + \mu_2(z) \equiv \text{constant}. \tag{23}$$

From this and the fact that f and g have the same zeros, we can deduce easily that

$$f(z) = c_1 e^{\beta(z)} + c_2 \tag{24}$$

and

$$g(z) = c(c_2 e^{-\beta(z)} + c_1), \tag{25}$$

where c and c_i $(i = 1, 2)$ are constants and $\beta(z)$ is an entire function of order less than one. This shows that f and g are related by relation (II).

Now we deal with subcase (ii), i.e., that $e^{\beta(z)-\alpha(z)} - 1 \not\equiv 0$. In this case, we have two possibilities: Case (a): $\beta(z) - \alpha(z) \not\equiv$ constant and Case (b): $\beta(z) - \alpha(z) \equiv$ constant $\neq 2n\pi i$ for any integer n. In Case (a) we have from (13) that

$$g'(z)\, h'(z) = \frac{[(\beta'(z) - \alpha'(z))\, e^{\beta(z)-\alpha(z)} - \alpha'(z)]}{e^{\alpha(z)}(e^{\beta(z)-\alpha(z)} - 1)}\, \alpha'(z). \tag{26}$$

We note that the left-hand side of Eq. (26) is an entire function. To insure that the function on the right-hand side of Eq. (26) also be entire, we must have the condition that whenever $e^{\beta(z)-\alpha(z)} - 1 = 0$, then either $\alpha'(z) = 0$ or $[\beta'(z) - \alpha'(z)]\, e^{\beta(z)-\alpha(z)} - \alpha'(z) = 0$. The latter case implies $\beta'(z) - 2\alpha'(z) = 0$. We are going to show that either $\alpha'(z)$ or $\beta'(z) - 2\alpha'(z)$ or both $\alpha'(z)$ and $\beta'(z) - 2\alpha'(z)$ must be identically zero. Suppose it were not so. Then by $\beta(z) - \alpha(z) \not\equiv$ constant it follows that the exponent of convergence of $e^{\beta(z)-\alpha(z)} - 1$ is at least one which is greater than both orders of $\beta'(z) - 2\alpha'(z)$ and $\alpha'(z)$ (since $\alpha(z)$ and $\beta(z)$ are assumed to be of order less than one and so are the orders of $\alpha'(z)$ and $\beta'(z)$ by [1]). Then the function on the right-hand side of identity (26) cannot be entire. Thus we must have $\alpha' \equiv 0$ or $\beta' - 2\alpha' \equiv 0$ or both are identically zero. All these cases have already been taken care of in the previous discussion.

Finally we treat Case (b): $\beta(z) = \alpha(z) \equiv c_0 =$ constant $\neq 2n\pi i$. In this case we have $e^{\beta(z)-\alpha(z)} \equiv$ constant $\neq 0$. From this and by taking the quotient of Eqs. (1) and (2), we have

$$\frac{f(z)}{g(z)} = \frac{f'(z)}{g'(z)}\, e^{\beta(z)-\alpha(z)} = e^{c_0}\, \frac{f'(z)}{g'(z)}\,. \tag{27}$$

Hence

$$\frac{f'(z)}{f(z)} = e^{c_0}\, \frac{g'(z)}{g(z)}\,.$$

This leads to

$$f(z) = cg(z)^k, \tag{28}$$

where c and k are two constants, which is relation (I). This completes the proof of the theorem.

3. Final Remark

It is not difficult to see that the theorem is still valid if one allows f and g to have multiple zeros but assumes that the exponents of convergence of these multiple zeros of f and g are less than $1 - \rho$ (ρ is the quantity defined in condition (C)).

4. Open Question

Suppose that two transcendental entire functions f and g assume the same zeros with the same multiplicities and that their first derivatives assume the same one-points with the same multiplicities. What can be said about the relationship between f and g?

References

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2. Chung-Chun Yang, On a Problem of Polynomials, to appear.